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SELF-SIMILAR PROBLEM OF CAUCHY-POISSON WAVES AT AN INCLINED SHORE

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We consider a two-dimensional problem concerning Cauchy-Poisson waves at an inclined shore in the case of an initial disturbance concentrated near the shore edge. We study the behavior of the solution near the shore and at large distances from it.

Numerous investigations, devoted to the study of standing and progressive waves on an inclined shore, are described in [1]. A two-dimensional problem concerning nonstationary waves on a shore with an angle of inclination $\gamma = \pi / 2n$, where n is an integer, was analyzed in [2, 3]. We consider below a case in which the angle of inclination is commensurable with $\pi/2$, subject to the condition that the initial disturbance is concentrated in the vicinity of the shore edge, so that the problem may be considered self-similar.

1. Two-dimensional nonstationary waves on the surface of a heavy fluid are determined from solving the following problem:

$$\Delta\varphi = 0 \tag{1.1}$$

$$g \frac{\partial\varphi}{\partial y} + \frac{\partial^2\varphi}{\partial t^2} = 0, \qquad y = 0$$

$$\frac{\partial\varphi}{\partial n} = 0, \qquad y = -x \operatorname{tg} \gamma$$

The coordinate origin here is located at a point of contact of the free surface with the shore, the x-axis is directed along the free surface, and the y-axis vertically upwards. As an initial condition, we assign the pressure distribution along the free surface (ρ_1 is the density)

$$\varphi(x, 0;\ 0) = P(x) / \rho_1, \qquad \partial\varphi(x, 0;\ 0) / \partial t = 0$$

More general initial conditions are also considered.

2. We consider the problem stated for the case in which the initial disturbance is concentrated near the origin and is characterized by a constant A which has the dimensionality $L^p T^q$.

We seek a solution of problem (1.1) in the form

$$\varphi(x, y;\ t) = A\rho^\alpha g^\beta \Phi(u, \theta), \quad u = \sqrt{g}t / 2\sqrt{\rho} \tag{2.1}$$

$$p + \alpha + \beta = 2, \qquad q - 2\beta = -1$$

Here ρ and θ are polar coordinates. The initial condition is formulated later. Substitution (2.1) into Eqs. (1.1) leads to the relations

$$4\alpha^2\Phi - 4\alpha u \quad \Phi_u' + u(u\Phi_u')_u' + 4\Phi_{\theta^2}'' = 0, \quad -\gamma < \theta < 0$$

$$4\Phi_\theta' + \Phi_{u^2}'' = 0, \quad \theta = 0; \quad \Phi_\theta' = 0, \quad \theta = -\gamma \tag{2.2}$$

We seek a solution of (2.2) in the form

$$\Phi(u,\theta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Phi_*(p,\theta)\,u^{-p}\,dp \quad (c=\mathrm{const}>0) \tag{2.3}$$

Substitution into Eqs. (2.2) gives

$$\begin{gathered}\Phi''_{*\,(\theta/2)^2}+(p+2\alpha)^2\,\Phi_*=0 \\ 4\partial\Phi_*(p,\theta)\,/\,\partial\theta+(p-1)(p-2)\,\Phi_*(p-2,\theta)=0,\quad \theta=0 \\ \partial\Phi_*(p,\theta)\,/\,\partial\theta=0,\qquad \theta=-\gamma\end{gathered} \tag{2.4}$$

The general solution of (2.4) may be represented in one of the following two forms:

$$\Phi_*(p,\theta)=U(p)\sin\left[\frac{p+2\alpha}{2}\theta+\sigma(p)\right] \tag{2.5}$$

$$\Phi_*(p,\theta)=U_1(p)\cos\frac{p+2\alpha}{2}\theta+U_2(p)\sin\frac{p+2\alpha}{2}\theta \tag{2.6}$$

Substituting (2.5) into the boundary conditions of (2.4), we obtain the relations

$$\begin{gathered}\sigma(p)=\frac{p\gamma}{2}+\alpha\gamma+\frac{\pi}{2}+k\pi \\ U(p+2)\sin\left[\frac{p\gamma}{2}+(\alpha+1)\gamma\right]-U(p)\frac{(p+1)p}{2(p+2+2\alpha)}\times \\ \cos\left[\frac{p\gamma}{2}+\alpha\gamma\right]=0\end{gathered} \tag{2.7}$$

Analogous relations, corresponding to (2.6), have the form

$$\begin{gathered}U_2(p)=-U_1(p)\,\mathrm{tg}\frac{p+2\alpha}{2}\gamma \\ U_1(p+2)-U_1(p)\frac{(p+1)p}{2(p+2+2\alpha)}\,\mathrm{ctg}\frac{p+2+2\alpha}{2}\gamma=0\end{gathered} \tag{2.8}$$

The second relation in (2.7) (or in (2.8)) is an equation in finite differences of the first order for determining the function $U(p)$ (or $U_1(p)$, respectively).

Further, we restrict ourselves to the special case $\alpha=-1$, $\beta=0$, corresponding to the initial impulse applied in the neighborhood of the coordinate origin. Other cases can be studied analogously; moreover, solutions corresponding to other α may be obtained from the solutions already found by differentiation and integration with respect to t.

The following functions satisfy the second of Eqs. (2.7) for $\gamma=\pi/2$ and $\gamma=\pi$, respectively:

$$\begin{gathered}U(p)=\omega(p)\,\Gamma\left(\frac{p+1}{2}\right),\qquad \gamma=\frac{\pi}{2} \\ U(p)=\omega(p)\,\Gamma\left(\frac{p+1}{2}\right)\left(\mathrm{ctg}\frac{p\pi}{2}\right)^{p/2}\cos^{-1}\frac{p\pi}{2},\ \gamma=\pi\end{gathered} \tag{2.9}$$

Here $\omega(p)$ is an arbitrary two-periodic function. Its choice signifies the selection of an initial condition for $\varphi(x, y;\ t)$ namely, the magnitude and plane of application of the impulse; in addition, the existence of the integral (2.3) must be guaranteed.

Let $\gamma=\pi m/2n$, where m and n are relatively prime integers. Here it is more suitable to use Eq. (2.6). The functions

$$U_1(p) = \omega(p)\,\Gamma\left(\frac{p+1}{2}\right)\Omega(p, m, n) \tag{2.10}$$

$$\Omega = \prod_{k=0}^{n-1} \sin \varkappa_k, \qquad m = 4l + 1$$

$$\Omega = \prod_{k=0}^{3n-1} \sin \varkappa_k, \qquad m = 4l + 3$$

$$\Omega = \prod_{k=0}^{n-1} \mathrm{ctg}^{(p+2k)/2n} \varkappa_k, \qquad m = 2l$$

$$\varkappa_k = (p + 2k)\frac{\pi}{4}\frac{m}{n}$$

satisfy the second equation of (2.8). Here $\omega(p)$ is also an arbitrary two-periodic function. Thus we have obtained a family of solutions of the initial problem, each of which corresponds to definite values of m and n. As we shall show below, the choice of the function $\omega(p)$ makes it possible to satisfy a wide class of initial conditions.

3. We pose the condition: the function $\Phi_*(p, \theta)$ (or the branch of it considered) must have at the point $p = 0$ a first order pole with residue proportional to $\sin\theta$. It is then found that for small θ

$$\varphi(\rho, \theta; 0) \sim \frac{\sin\theta}{\rho} \approx \frac{y}{x^2 + y^2}$$

It is obvious [4] that for $y \to 0$ the function $y / \pi(x^2 + y^2)$ tends to δ-function, so that the distribution along the free surface of the initial impulse is proportional to $\delta(\rho)$.

The final solution of the problem concerning Cauchy-Poisson waves, in the case of an impulse applied to the free surface for $\gamma = \pi/2$ and $\gamma = \pi$, is obtained from Eqs. (2.9) by choosing for $\omega(p)$ the functions $(2\sqrt{\pi})^{-1}\,\mathrm{ctg}\, p\pi/2$ and $(2\sqrt{\pi})^{-1}$ $\mathrm{ctg}^2\, p\pi/2$, respectively. It should be noted that for other γ values the solutions, which are found in the form (2.5), correspond to initial impulses applied in planes inclined to the horizontal. For example, suppose that for $\gamma = \pi/2$ in (2.9) we choose $\omega = (2\sqrt{\pi})^{-1}\,\mathrm{ctg}^2 p\pi/2$. Then for $p = 0$ the residue is proportional to $\cos\theta = \cos(\pi/2 - \theta_1) = \sin\theta_1$, i.e., the impulse is applied to the free surface at a right angle.

If it be required to solve the problem for an arbitrary $\gamma = m\pi/2n$, in the case of an impulse applied to the free surface or at an arbitrary given angle to it, it is sufficient to construct a linear combination of two solutions of the form (2.5), which differ by a factor having for $\rho = 0$ a first order pole, the coefficients being chosen from the condition that the residue at the point $p = 0$ has a given form.

For example, let $\gamma = 3\pi/4$. Transforming the corresponding expression from (2.10) in the manner described, we can obtain the following solution of the initial problem satisfying the required conditions for $t = 0$ on the free surface:

$$\Phi_*(p, \theta) = \left(\mathrm{ctg}\frac{p\pi}{2} + \beta_1\,\mathrm{ctg}^2\frac{p\pi}{2}\right)\sin^{-2} p\pi\,\Gamma\left(\frac{p+1}{2}\right)\prod_{k=0}^{4}\sin\varkappa_k \sin\left(\varkappa_5 + \frac{p\theta}{2} - \theta\right)$$

$$\beta_1 = [{}^9/_4 - \psi({}^1/_2)/\pi]^{-1} \tag{3.1}$$

Here $\psi(p)$ is the logarithmic derivative of the Γ-function and the factor $\sin^{-2} p\pi$ is introduced in order that the intergral (2.3) will exist. By completing the contour of integration in (2.3) so as to form a closed semicircle in the left halfplane and applying the theorem on residues, we can obtain a representation [5] of the solution in the form of a series for $\gamma = \pi/2$

$$\Phi(u,\theta) = \frac{-1}{\pi^{3/2}} \sum_{n=0}^{\infty} \Gamma\left(-n+\frac{1}{2}\right) u^{2n} \sin\left[\frac{n\pi}{2} + (n+1)\theta\right]$$

It is known from [5] that in problems of the type considered it is sufficient, for a numerical analysis of the solutions, to find, in practically important cases, asymptotic expressions for the integrals at large values of u. Below we derive corresponding formulas for certain γ values when $\theta = 0$. In other cases the calculations are similar.

Estimates show that for the solutions obtained the absolute value of the integrand function in Eq. (2.3) increases on lines parallel to the imaginary axis when $c > 0$ and decreases when $c < 0$ for an increase in $|\mathrm{Im}\ p|$, and that it has a minimum on the imaginary axis for large $|p|$. Therefore, in calculating the integral (2.3) for large u, it is natural to take the contour as shown in Fig. 1, passing through a saddle point, which is to be found for large $|p|$

Let $\gamma = \pi/2$. The integral (2.3) can be written in the form

$$\Phi(u, 0) = \frac{1}{4\pi^{3/2} i}\left[\int_c^{-c+i\infty} L(u, p)\, dp + \int_{-c-i\infty}^{c} L(u, p)\, dp\right] = I_1 + I_2$$

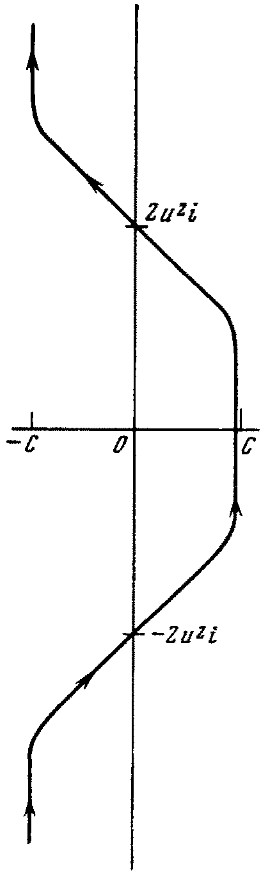

Fig. 1

$$L(u, p) = \Gamma\left(\frac{p+1}{2}\right)(e^{ip\pi/2} + e^{-ip\pi/2})\,(e^{ip\pi/4} + e^{-ip\pi/4})^{-1}$$

For large $|p|$ we can neglect small terms in I_1 and I_2 and we can use the asymptotic representation for $\Gamma[(p+1)/2)]$. Then (2.3), by substitution of the variables $p = 2u^2 t$, assumes the form

$$\Phi(u, 0) = \frac{u^2}{2\sqrt{2\pi i}}\left\{\int_c^{-c+i\infty} \exp\left[u^2\left(t\ln t - t - it\,\frac{\pi}{2}\right)\right] dt + \int_{-c-i\infty}^{c} \exp\left[u^2\left(t\ln t - t + it\,\frac{\pi}{2}\right)\right] dt\right\}$$

The saddle points are located at i and $-i$, respectively. Calculations show that the steepest descent directions make angles of $\pm\pi/4$ with the imaginary axis. The usual saddle method formulas lead to the following asymptotic expression [5]:

$$\varphi(x,\,0;\,t)=\frac{Ag^{1/2}t}{2\,\pi^{1/2}\rho_1 x^{3/2}}\sin\left(\frac{gt^2}{4x}+\frac{\pi}{4}\right)$$

Passing to the case $\gamma=\pi$, we note that although the function $\Phi_*\,(p,\,\theta)$ is multi-valued, the same reasoning applies as in the case just analyzed since we can consider its single-valued branch. The integral (2.3) assumes the form

$$\Phi(u,\,0)=-\frac{1}{4\pi^{3/2}i}\int\limits_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{p+1}{2}\right)\operatorname{ctg}^{p/2+2}\frac{p\pi}{2}\,u^{-p}\,dp$$

An analogous decomposition of the integral into two parts, the use of asymptotic expressions for the integrand function, the same substitution of variables, and the application of the saddle point method lead to the asymptotic expression

$$\varphi(x,\,0;\,t)=\frac{Ag^{1/2}t}{2\pi^{1/2}\rho_1 x^{3/2}}\sin\left(\frac{gt^2}{4x}+\frac{\pi}{4}+\frac{2}{\pi}\right)$$

In the case $\gamma=3\pi\,/\,4$, analogous considerations yield the formula

$$\varphi(x,\,0;\,t)\approx\frac{Ag^{1/2}t}{3\pi^{1/2}\rho_1 x^{3/2}}\cos\frac{gt^2}{4x}$$

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CONDITIONS FOR SHOCKLESS STATE OF THE VORTEX FLOW IN TWO-DIMENSIONAL LAVAL NOZZLES

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Conditions given in [1, 2] for the absence of shocks in the flow in the vicinity of the center of a nozzle for two-dimensional vortex-free flows of an ideal gas are generalized to the case of rotational flows. Both continuous flows and flows with shock waves are constructed.

1. We take the origin of a Cartesian system of coordinates at the nozzle center, with the x-axis directed along the axis of the nozzle and the y-axis perpendicular to it. We assume that in the neighborhood of the nozzle center the entropy $S\,(y)$ is a